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Rational Approximation with Varying Weights III

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Approximation by weighted rationals of the form $w^n r_n$, where $r_n = p_n/q_n$, p_n and q_n are polynomials of degree at most $[\alpha n]$ and $[\beta n]$, respectively, and w is an admissible weight, is investigated on compact subsets of the real line for a general class of weights and given $\alpha \geqslant 0$, $\beta \geqslant 0$, with $\alpha + \beta > 0$. Conditions that characterize the largest sets on which such approximation is possible are given. We apply the general theorems to Laguerre and Freud weights.

1. MAIN RESULTS

The problem of uniform approximation on compact subsets of the real line by weighted rational functions of the form $w^n r_n$, where w is an admissible weight, and r_n is a rational function, was investigated in [1, 7]. Here we further generalize the previous results and we consider applications to Laguerre and Freud weights.

For $n \in \mathbf{N}$, let $\mathscr{P}_n$ denote the space of algebraic polynomials of degree at most n. For a compact set E, $C(E)$ denotes the space of continuous real-valued functions on E. The symbol $[\cdot]$ denotes the greatest integer function.

Let Σ be a closed regular subset of the real line $\mathbf{R}$ and $w: \Sigma \to [0, \infty)$ be a *strongly admissible weight*, that is, w is continuous on Σ, it is positive on a set of positive capacity, and if Σ contains a neighborhood of the point ∞, then $|x|\, w(x) \to 0$ as $|x| \to \infty$, $x \in \Sigma$. Let $\alpha \geqslant 0$, $\beta \geqslant 0$ with $\alpha + \beta > 0$ be given numbers.

We shall first consider the problem of characterizing the compact sets $E \subseteq \Sigma$ having the approximation property that every function $f \in C(E)$ is the uniform limit on E of a sequence $\{w^n r_n\}_{n=1}^{\infty}$, with $r_n = p_n/q_n$, $p_n \in \mathscr{P}_{[\alpha n]}$ and $q_n \in \mathscr{P}_{[\beta n]}$.

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From previous results from [10, 8] regarding weighted polynomial approximation (the case $\alpha = 1$, $\beta = 0$) it is known that $E \subseteq S_w$, where S_w is the support of an extremal measure μ_w, the unique probability measure that minimizes the weighted energy

$$I_w(\mu) := \iint \log \frac{1}{|z-t|\, w(z)\, w(t)}\, d\mu(z)\, d\mu(t)$$

over the set $\mathcal{M}(\Sigma)$ of all probability Borel measures μ supported on Σ. It is also known [6, 8], that S_w is a compact set, and the following representation for $w(x)$ holds on S_w:

$$w(x) = \exp(U^{\mu_w}(x) - F_w), \qquad x \in S_w, \tag{1.1}$$

where F_w is a constant, and for a compactly supported Borel measure μ the logarithmic potential U^{μ} is defined by

$$U^{\mu}(z) := \int \log \frac{1}{|z-t|}\, d\mu(t), \qquad z \in \mathbf{C}.$$

In [7, Theorem 1.5], it was shown that representation like (1.1) on an interval I with μ_w replaced by a signed measure $\mu = \mu^+ - \mu^-$ with absolutely continuous $\mu^{\pm}$ having densities that behave like $|t-c|^{-1/2}$ at the endpoints c of I allows approximation on I. Thus the largest set E having the approximation property is essentially the largest set E on which w can be written as the exponential of the logarithmic potential of an absolutely continuous signed measure.

Before stating the main results of the paper we introduce some notation.

Let $K \subset \mathbf{R}$ be a compact set of positive logarithmic capacity and ω_K be its equilibrium measure, that is, the measure which minimizes the unweighted logarithmic energy $I_1(\mu)$ over all measures $\mu \in \mathcal{M}(K)$. If ω_K is absolutely continuous with respect to Lebesgue measure, then by f_K we shall denote its density.

Let v be a positive measure supported on K. For $y \in \mathbf{R}$ we define the signed measure

$$\sigma(y) := v - y\omega_K.$$

Let $\sigma(y) = \sigma^+(y) - \sigma^-(y)$ be the Jordan decomposition of $\sigma(y)$ and set

$$p(y) := \|\sigma^+(y)\| \qquad \text{and} \qquad n(y) := \|\sigma^-(y)\|.$$

Our first theorem combines and extends Theorem 1 of [1] and Theorem 1.5 of [7].

THEOREM 1.1. *Let w be a strongly admissible weight defined on a set E which is the union of finitely many closed intervals. Let $\alpha \geqslant 0$, $\beta \geqslant 0$ with $\alpha+\beta>0$ be given numbers. Assume that*

$$w(u)=\exp(U^{\nu}(u)+F), \qquad u \in E, \tag{1.2}$$

where F is a constant, $d\nu(t)=v(t)\,dt$ on E, and the density v is continuous and nonnegative on $\operatorname{Int}(E)$, *and at each endpoint c of E,*

$$v(t)\,|t-c|^{1/2} \to l_c < \infty, \qquad t \to c, \quad t \in E. \tag{1.3}$$

Let $\sigma(y)=\nu-y\omega_E$.

First assume that there is a sequence of weighted rationals of the form $w^n p_n/q_n$ with $p_n \in \mathcal{P}_{[\alpha n]}$ and $q_n \in \mathcal{P}_{[\beta n]}$ such that $w^n p_n/q_n \to 1$ as $n \to \infty$ uniformly on E. Then there exists $y \in \mathbf{R}$ with $p(y) \leqslant \alpha$ and $n(y) \leqslant \beta$.

Next assume that there exists $y \in \mathbf{R}$ such that $p(y)<\alpha$ and $n(y)<\beta$. Then every function $f \in C(E)$ is the uniform limit on E of a sequence of weighted rationals $\{w^n p_n/q_n\}_{n=1}^{\infty}$ with $p_n \in \mathcal{P}_{[\alpha n]}$ and $q_n \in \mathcal{P}_{[\beta n]}$.

Remark 1.2. If E is a compact set with $p(x)=\alpha$ and $n(x)=\beta$ for some x, then weighted rational approximation on E of functions not vanishing on E is not always possible. This is the case for the exponential weight $w(u)=e^u$ on the interval $[0, 2\pi]$. In this case, for $\alpha=\beta=1$, A. B. J. Kuijlaars has proved that every function $f \in C([0, 2\pi])$ that has at least one zero on $[\pi, 2\pi]$ is approximable. Hence neither of the conditions of Theorem 1.1 is both necessary and sufficient.

It turns out that for certain classes of admissible weights w the conditions of Theorem 1.1 are satisfied on each set S_{w^λ}, $\lambda>0$.

COROLLARY 1.3. *Let* $w(u)=\exp(-Q(u))$ *be a positive continuous weight defined on a set $\Sigma \subset \mathbf{R}$ that is the union of finitely many closed intervals $\{I_j\}_{j=1}^{m}$. Assume that on each interval I_j, the external field $Q(u)$ is convex or $|u|\,Q'(u)$ is increasing, and for some $p \in (1, \infty)$, $w' \in L^p(\Sigma)$. Then w satisfies the conditions of Theorem* 1.1. *Furthermore, Theorem* 1.1 *holds on each set $E=S_{w^\lambda}$, $\lambda>0$.*

Conversely, if w is a weight satisfying the conditions of Theorem 1.1 *on a set E that is the union of finitely many intervals, then $E=S_{w^\lambda}$ for some $\lambda>0$.*

COROLLARY 1.4. *Let* $w(u)=\exp(-Q(u))$ *be an admissible weight defined on a set $\Sigma \subset \mathbf{R}$ that is the union of finitely many intervals, and assume that Q is a real-analytic function on Σ. Then w satisfies the conditions of Theorem* 1.1. *Furthermore, Theorem* 1.1 *holds on each set $E=S_{w^\lambda}$, $\lambda>0$.*

The above results and the representation

$$w(x) = \exp(U^{(1/\lambda)\,\mu_{w^\lambda}}(x) - (1/\lambda)\, F_{w^\lambda}), \qquad x \in S_{w^\lambda} \tag{1.4}$$

which follows from (1.1) suggest that it is important to study weighted rational approximation on the sets S_{w^λ}. Before we state the corresponding approximation problem we mention that by [8, Theorem IV.4.1], ([10, Lemma 5.4])

$$S_{w^{\lambda_1}} \subseteq S_{w^{\lambda_2}}, \qquad \text{for} \quad \lambda_1 > \lambda_2 > 0. \tag{1.5}$$

Now we state the first approximation problem:

(A1) *For given $\alpha \geqslant 0$ and $\beta \geqslant 0$ with $\alpha + \beta > 0$ find the largest set S_{w^λ} (or, equivalently, the smallest $\lambda > 0$) with the property that on every compact set $E \subset \mathrm{Int}(S_{w^\lambda})$ every function $f \in C(E)$ is the uniform limit on E of a sequence $\{w^n p_n/q_n\}_{n=1}^{\infty}$ with $p_n \in \mathcal{P}_{[\alpha n]}$ and $q_n \in \mathcal{P}_{[\beta n]}$.*

Before stating the next theorem we introduce some notation. For $\lambda > 0$ and $y \in \mathbf{R}$ we define the signed measure

$$\sigma_\lambda(y) := \frac{1}{\lambda}\, \mu_{w^\lambda} - y\omega_\lambda, \tag{1.6}$$

where $\omega_\lambda := \omega_{S_{w^\lambda}}$, and we set

$$p_\lambda(y) := \|\sigma_\lambda^+(y)\|, \qquad n_\lambda(y) := \|\sigma_\lambda^-(y)\|. \tag{1.7}$$

THEOREM 1.5. *Let w be a strongly admissible weight defined on a closed and regular set $\Sigma \subseteq \mathbf{R}$. Assume that for every $\lambda > 0$, S_{w^λ} is the union of finitely many closed intervals, the extremal measure μ_{w^λ} is absolutely continuous on S_{w^λ}, its density v_λ is continuous and nonnegative on* $\mathrm{Int}(S_{w^\lambda})$, *and at each endpoint c of S_{w^λ},*

$$v_\lambda(t)\, |t - c|^{1/2} \to l_\lambda(c) < \infty, \qquad t \to c, \quad t \in S_{w^\lambda}. \tag{1.8}$$

Assume further that $S_{w^{\lambda_1}} \subset S_{w^{\lambda_2}}$ for all $\lambda_1 > \lambda_2 > 0$. In particular this is true if $Q = \log(1/w)$ is real-analytic on Σ, and $v_\lambda(c) = 0$ at each endpoint c of S_{w^λ} for all $\lambda > 0$.

Then the infimum of all numbers $\lambda > 0$ such that on every compact set $E \subset S_{w^\lambda}$ every function $f \in C(E)$ is the uniform limit on E of a sequence $\{w^n p_n/q_n\}_{n=1}^{\infty}$ with $p_n \in \mathcal{P}_{[\alpha n]}$ and $q_n \in \mathcal{P}_{[\beta n]}$ is the number $\lambda^* = \lambda^*(\alpha, \beta)$ defined by

$$\lambda^*(\alpha, \beta) = \inf\{\lambda > 0 : \exists\, y \in \mathbf{R} : p_\lambda(y) < \alpha,\ n_\lambda(y) < \beta\}, \tag{1.9}$$

if $\alpha > 0$ *and* $\beta > 0$, *and*

$$\lambda^*(\alpha, 0) = \inf\{\lambda > 0 : \exists y \in \mathbf{R} : p_\lambda(y) < \alpha,\, n_\lambda(y) = 0\}, \tag{1.10}$$

$$\lambda^*(0, \beta) = \inf\{\lambda > 0 : \exists y \in \mathbf{R} : p_\lambda(y) = 0,\, n_\lambda(y) < \beta\}. \tag{1.11}$$

Finally we consider approximation by weighted rationals $w^n p_n/q_n$ with $p_n \in \mathcal{P}_{[\alpha n]}$, $q_n \in \mathcal{P}_{[\beta n]}$ for $\alpha \geqslant 0$ and $\beta \geqslant 0$ with a positive sum $\alpha + \beta$ that does not exceed a given number $\gamma > 0$.

Let w be a strongly admissible weight defined on a closed and regular set $\Sigma \subset \overline{\mathbf{R}}$ and assume that w satisfies the conditions of Theorem 1.5. The second approximation problem is stated below:

(A2) *For given* $\gamma > 0$ *find the largest set* S_{w^λ} *(equivalently find the smallest* $\lambda > 0$*) such that there exist* $\alpha \geqslant 0$ *and* $\beta \geqslant 0$ *with* $\alpha + \beta \in (0, \gamma]$ *having the property that on every compact set* $E \subset \operatorname{Int}(S_{w^\lambda})$ *every function* $f \in C(E)$ *is the uniform limit of a sequence of weighted rationals* $\{w^n p_n/q_n\}_{n=1}^{\infty}$ *with* $p_n \in \mathcal{P}_{[\alpha n]}$, $q_n \in \mathcal{P}_{[\beta n]}$.

For $\lambda > 0$ and $y \in \mathbf{R}$ we set $m_\lambda(y) := p_\lambda(y) + n_\lambda(y)$, where $p_\lambda(y)$ and $n_\lambda(y)$ are defined in (1.7) and (1.6). Then $m_\lambda(y) \leqslant 1/\lambda + |y|$ and

$$m_\lambda(y) = \int |d\sigma_\lambda(y)| = \int |(1/\lambda)\, d\mu_{w^\lambda} - y\, d\omega_{S_{w^\lambda}}|$$

$$\geqslant \left| \int |(1/\lambda)\, d\mu_{w^\lambda}| - \int |y\, d\omega_{S_{w^\lambda}}| \right| = |1/\lambda - |y||. \tag{1.12}$$

From these inequalities we get

$$m_\lambda := \inf\{m_\lambda(y) \colon y \in \mathbf{R}\} = \min\{m_\lambda(y) \colon y \in [0, 2/\lambda]\}. \tag{1.13}$$

Let f_λ be the equilibrium density for the set S_{w^λ}, and

$$s_\lambda(t, y) := \frac{1}{\lambda} v_\lambda(t) - y f_\lambda(t)$$

be the density of the signed measure $\sigma_\lambda(y)$.

THEOREM 1.6. *Let* $\gamma > 0$ *be given and* w *satisfy the conditions of Theorem* 1.5. *Assume that for every* $\lambda > 0$ *and* $y \in \mathbf{R}$, $s_\lambda(t, y)$ *has at most countably many zeros in* S_{w^λ}. *Then the largest set* S_{w^λ} *having the property that for every compact* $E \subset \operatorname{Int}(S_{w^\lambda})$ *every function* $f \in C(E)$ *is the uniform limit on* E *of a sequence of weighted rationals* $\{w^n p_n/q_n\}_{n=1}^{\infty}$ *with* $p_n \in \mathcal{P}_{[\alpha n]}$ *and*

$q_n \in \mathscr{P}_{[\beta n]}$ for some $\alpha \geqslant 0$ and $\beta \geqslant 0$ with $\alpha + \beta \in (0, \gamma]$, is the set $S_{w^{\lambda(\gamma)}}$, where $\lambda(\gamma) \in (0, 1]$ is the solution of the equation

$$m_\lambda = \gamma. \tag{1.14}$$

2. PROOFS

Proof of Theorem 1.1. The proof of the necessity part of the theorem is the same as the proof of Lemma 5 of [1].

The sufficiency part follows from Theorem 1.5 in [7] and Lemma 4.4 in [9]. It is known that for a set $E = \bigcup_{j=1}^m [a_j, b_j]$ with $a_1 < b_1 < a_2 < \cdots < a_m < b_m$ the equilibrium distribution ω_E has the form

$$d\omega_E(t) = f_E(t)\, dt = \frac{|S(t)|}{\pi \sqrt{|R(t)|}}\, dt, \qquad t \in E, \tag{2.1}$$

where

$$R(t) = \prod_{j=1}^{m} ((t - a_j)(t - b_j)) \qquad \text{and} \qquad S(t) = \prod_{j=1}^{m-1} (t - y_j)$$

for some $y_j \in (b_j, a_{j+1})$, $j = 1, ..., m-1$ (see, for example, [9, Lemma 4.4]). From (2.1) we see that at the endpoints of E, the density $f_E(t)$ has the same behavior as the density $v(t)$ and the result follows from Theorem 1.5 of [7]. ∎

Proof of Corollary 1.3. Since $w^\lambda = \exp(-\lambda Q)$, for $\lambda > 0$ the external field for w^λ has the same properties as Q. Hence it is enough to consider w only. By [8, Theorem IV.1.10(d)], the support S_w is the union of intervals $\{J_k\}$ at most one lying in any of the intervals I_j (the components of Σ). Furthermore, if J is one of the intervals J_k, by Theorem IV.1.6(e) of [8] we have

$$\mu_{w|_J} = \mu_w|_J + \overline{\mu_w|_{(\mathbf{R} \setminus J)}},$$

where the bar denotes taking balayage onto J out of $\mathbf{C} \setminus J$. This implies that $S_{w|_J} = J$. Then by [8, Theorem IV.2.4] applied to $w|_J$, and by [8, Corollary II.4.12] according to which the measure $\overline{\mu_w|_{(\mathbf{R} \setminus J)}}$ has continuous density it follows that (1.3) holds for the density of $\mu_{w|_J}$. The representation (1.2) for w on S_w follows from (1.1).

Now suppose that E is the union of finitely many intervals, and w satisfies the conditions of Theorem 1.1 on E. In particular $w(u) = \exp(U^v(u) + F)$, $u \in E$, with density $v = v^+ - v^-$ satisfying (1.3). From (2.1) and (1.3) we get that for $\gamma > 0$ large enough ($\gamma > \sup\{v^-(t)/f_E(t), t \in E\}$), $v_1 := v + \gamma f_E > 0$ on E. Setting $\lambda := (\|v^+\| + \gamma - \|v^-\|)^{-1} > 0$ and $F_1 := \lambda(F - \gamma \log(1/\mathrm{cap}(E)))$ we obtain

$$w^\lambda(u) = \exp(U^{\lambda v_1}(u) + F_1), \qquad u \in E,$$

and then by [8, Theorem I.3.3] we get $S_{w^\lambda} = E$ and $\mu_{w^\lambda} = \lambda v_1 \, dt$. ∎

Proof of Corollary 1.4. For a real-analytic external field Q it was shown in [3, Theorem 38] that S_w is the union of finitely many closed intervals, the measure μ_w is absolutely continuous on S_w, and its density satisfies the conditions of Theorem 1.1. The same is true for w^λ for any $\lambda > 0$. Thus the corollary follows from Theorem 1.1. ∎

For the proof of Theorem 1.5 we need a lemma.

LEMMA 2.1. *Let $E_1 \subset E_2$ be compact sets on the real line. Assume that each E_j, $j = 1, 2$ is the union of finitely many intervals. Let $\omega_j = \omega_{E_j}$ and f_j denote the equilibrium measure and density for E_j, respectively. Then $f_1 \geqslant f_2$ on E_1 and for every interval $I \subseteq E_1$,*

$$\omega_1(I) > \omega_2(I).$$

Proof. By Lemma 5.5 of [10] (or [8, Theorem IV.1.6(e)]) we have

$$\omega_1 = \overline{\omega_2} = \omega_2|_{E_1} + \overline{\omega_2|_{E_2 \setminus E_1}} \geqslant \omega_2|_{E_1},$$

where the bar denotes taking balayage onto E_1 out of $\mathbf{C} \setminus E_1$. Thus $f_1 \geqslant f_2$ on E_1. We set $v := \omega_2|_{E_2 \setminus E_1}$.

Now assume that there is an interval $I \subset E_1$ such that $\omega_1(I) = \omega_2(I)$. Then $\bar{v}(I) = 0$. Let h be a continuous function on E_1 that vanishes on $E_1 \setminus I$ and is positive on $\mathrm{Int}(I)$, and let H denote the solution of the Dirichlet problem on the domain $D = \overline{\mathbf{C}} \setminus E_1$ with boundary function h (see [8, Section I.2]). This function H is harmonic on D and continuous on $\overline{\mathbf{C}}$, and by the minimum principle, [8, Theorem I.2.4], it is also positive on D. By a property of balayage measures, [8, Theorem II.4.1(c)], we have

$$\int H \, d\bar{v} = \int H \, dv$$

which is a contradiction. Indeed the left integral is $\int_{E_1} h \, d\bar{v} = 0$ by the choice of h, and the right integral is positive since it is over $E_2 \setminus E_1 \subset D$ where $H > 0$ and by (2.1) $v' = f_2 > 0$. ∎

Proof of Theorem 1.5. We assume that $\alpha > 0$ and $\beta > 0$, the proof in the other two cases is similar.

First let $\lambda > \lambda^*$ and $E \subset S_{w^\lambda}$ be a compact set. Let $f \in C(E)$ and $f_1 \in C(S_{w^\lambda})$ be an extension of f to S_{w^λ}. Then there is $y \in \mathbf{R}$ such that $p_\lambda(y) < \alpha$ and $n_\lambda(y) < \beta$, and by (1.4) and Theorem 1.1, f_1 is uniformly approximable on S_{w^λ} by weighted rationals $w^n p_n/q_n$ with $p_n \in \mathcal{P}_{[\alpha n]}$, $q_n \in \mathcal{P}_{[\beta n]}$ and so is f on E.

When $\lambda = \lambda^*$ we can verify the approximation property only on compact sets $E \subset \operatorname{Int}(S_{w^{\lambda^*}})$. Indeed let E be such set. By Lemma 5.8 of [10] for every $x \in \operatorname{Int}(S_{w^{\lambda^*}})$ there is a $\lambda(x) > \lambda^*$ such that $x \in \operatorname{Int}(S_{w^{\lambda(x)}})$. Then

$$E \subset \bigcup_{x \in E} \operatorname{Int}(S_{w^{\lambda(x)}})$$

and since E is compact there is a finite subcover of E, $\{\operatorname{Int}(S_{w^{\lambda(x_i)}})\}_{i=1}^{k(E)}$. Let $\lambda := \min\{\lambda(x_i)\colon 1 \leqslant i \leqslant k(E)\}$. Then $\lambda > \lambda^*$, $E \subset S_{w^\lambda}$, and as we have already shown every $f \in C(E)$ is uniformly approximable on E by weighted rationals $w^n p_n/q_n$ with $p_n \in \mathcal{P}_{[\alpha n]}$ and $q_n \in \mathcal{P}_{[\beta n]}$.

In verifying the converse it is enough to assume that $S_{w^{\lambda_1}} \subset S_{w^{\lambda_2}}$ for all $\lambda_1 > \lambda_2 > 0$. Indeed in the case when w is real-analytic on Σ we have by Lemma 2.3 of [2] for every $\lambda_0 > 0$,

$$\bigcup_{\lambda > \lambda_0} S_{w^\lambda} = \{t \in S_{w^{\lambda_0}} : v_{\lambda_0}(t) > 0\},$$

and since for every $\lambda > 0$, v_λ vanishes at the endpoints of S_{w^λ} we get $S_{w^{\lambda_1}} \subset S_{w^{\lambda_2}}$ for $\lambda_1 > \lambda_2 > 0$. By Theorem IV.4.9 of [8] (or Lemma 5.7 of [10]) with $w := w^{\lambda_2}$, $\lambda := \lambda_1/\lambda_2 > 1$, and $d\mu_{w^\lambda} = v_\lambda\, dt$ we have

$$\frac{\lambda_2}{\lambda_1} v_{\lambda_1} \leqslant v_{\lambda_2}|_{S_{w^{\lambda_1}}} - \left(1 - \frac{\lambda_2}{\lambda_1}\right) f_{S_{w^{\lambda_2}}}|_{S_{w^{\lambda_1}}}. \tag{2.2}$$

If $\lambda^* = 0$ there is nothing to prove. So assume that $\lambda^* > 0$ and let $\lambda \in (0, \lambda^*)$. In view of Theorem 1.1 it is enough to show that for all $y \in \mathbf{R}$

$$h_\lambda(y) := \max\{p_\lambda(y), (\alpha/\beta)\, n_\lambda(y)\} > \alpha.$$

Indeed assume that there is y_0 with $h_\lambda(y_0) = \alpha$. By definition $p_\lambda(y) \geqslant 0$ is a decreasing function of y and $p_\lambda(0) = 1/\lambda > 0$. Similarly $n_\lambda(y) \geqslant 0$ is an increasing function of y and $n_\lambda(0) = 0$. Hence $h_\lambda := \inf\{h_\lambda(y)\colon y \in \mathbf{R}\}$ is attained at unique $y_\lambda > 0$ and

$$h_\lambda = p_\lambda(y_\lambda) = (\alpha/\beta)\, n_\lambda(y_\lambda). \tag{2.3}$$

Since $\lambda < \lambda^*$ from the definition of λ^* we get

$$\alpha = h_\lambda(y_0) \geqslant h_\lambda = \min\{h_\lambda(y) \colon y \in \mathbf{R}\} \geqslant \alpha,$$

that is, $h_\lambda(y_0) = h_\lambda(y_\lambda) = \alpha$. Then $y_0 = y_\lambda > 0$ and $p_\lambda(y_0) = (\alpha/\beta)\, n_\lambda(y_0) = \alpha$.

Let $\lambda_1 \in (\lambda, \lambda^*)$. By (2.2) with $\lambda_2 = \lambda$ and Lemma 2.1, for $y > 0$ such that $p_{\lambda_1}(y) > 0$ we have

$$\begin{aligned} \frac{1}{\lambda_1} v_{\lambda_1} - y f_{S_{w^{\lambda_1}}} &\leqslant \frac{1}{\lambda} v_\lambda |_{S_{w^{\lambda_1}}} - \left(\frac{1}{\lambda} - \frac{1}{\lambda_1}\right) f_{S_{w^\lambda}} |_{S_{w^{\lambda_1}}} - y f_{S_{w^{\lambda_1}}} \\ &\leqslant \frac{1}{\lambda} v_\lambda |_{S_{w^{\lambda_1}}} - \left(\frac{1}{\lambda} - \frac{1}{\lambda_1} + y\right) f_{S_{w^\lambda}} |_{S_{w^{\lambda_1}}}. \end{aligned} \tag{2.4}$$

We integrate (2.4) over $\operatorname{supp}(\sigma^+_{\lambda_1}(y))$. Since $S_{w^{\lambda_1}} \subset S_{w^\lambda}$, applying Lemma 2.1 we obtain

$$p_{\lambda_1}(y) < p_\lambda(y + 1/\lambda - 1/\lambda_1). \tag{2.5}$$

We set $y = y_0 - 1/\lambda + 1/\lambda_1 > 0$ for $\lambda_1 \in (\lambda, \lambda^*)$ close enough to λ (so that $1/\lambda - 1/\lambda_1 < y_0/2$), and we obtain

$$p_{\lambda_1}(y_0 - 1/\lambda + 1/\lambda_1) < p_\lambda(y_0) = \alpha.$$

Then using the identity $p_\lambda(y) - n_\lambda(y) = 1/\lambda - y$ we obtain

$$\begin{aligned} n_{\lambda_1}(y_0 - 1/\lambda + 1/\lambda_1) &= p_{\lambda_1}(y_0 - 1/\lambda + 1/\lambda_1) + y_0 - 1/\lambda \\ &< p_\lambda(y_0) + y_0 - 1/\lambda = n_\lambda(y_0) = \beta. \end{aligned}$$

We get $h_{\lambda_1}(y_0 - 1/\lambda + 1/\lambda_1) < \alpha$ which contradicts the choice of $\lambda_1 < \lambda^*$. Moreover, we have shown that h_λ is a decreasing function of $\lambda > 0$.

We proved that if $\lambda \in (0, \lambda^*)$ then $h_\lambda(y) > \alpha$ for all $y \in \mathbf{R}$.

Let $\lambda \in (0, \lambda^*)$ and let $E = S_{w^{\lambda_1}}$ for some $\lambda_1 \in (\lambda, \lambda^*)$. Then the function 1_E (the characteristic function of the set E) is not uniformly approximable on E by weighted rationals $w^n p_n/q_n$ with $p_n \in \mathcal{P}_{[\alpha n]}$ and $q_n \in \mathcal{P}_{[\beta n]}$, because, otherwise we would have by Theorem 1.1 an $y \in \mathbf{R}$ with $h_\lambda(y) \leqslant \alpha$ and as we have shown this is impossible.

Theorem 1.5 is proved. ∎

For the proof of Theorem 1.6 we will need the following lemma.

LEMMA 2.2. *Assume that for every $\lambda > 0$ and $y \in \mathbf{R}$ the density $s_\lambda(t, y)$ of the signed measure $\sigma_\lambda(y)$ has at most countably many zeros in S_{w^λ}. Then the function $m_\lambda(y) \in C^1(\mathbf{R})$ and there is a unique $y^* = y^*(\lambda)$ such that $m_\lambda = m_\lambda(y^*)$.*

Proof. Let $s_\lambda^\pm(t, y)$ be the densities of $\sigma_\lambda^\pm(y)$ respectively. It follows from the representation

$$m_\lambda(y)=\int_{S_{w^\lambda}} |(1/\lambda)\, d\mu_{w^\lambda}(t)-y\, d\omega_\lambda(t)|$$

that $m_\lambda(y)\in C(\mathbf{R})$. Let $y_0\in\mathbf{R}$ be fixed. By the definition of $m_\lambda(y)$,

$$\begin{aligned}
m_\lambda(y)-m_\lambda(y_0)&\\
&=\int_{S_{w\lambda}} (s_\lambda^+(t, y)-s_\lambda^+(t, y_0))\, dt+\int_{S_{w\lambda}} (s_\lambda^-(t, y)-s_\lambda^-(t, y_0))\, dt\\
&=2\int_{S_{w\lambda}} (s_\lambda^+(t, y)-s_\lambda^+(t, y_0))\, dt-\int_{S_{w\lambda}} (s_\lambda(t, y)-s_\lambda(t, y_0))\, dt\\
&=(y-y_0)-2\int_{\Delta_\lambda^+(y)\cap\Delta_\lambda^+(y_0)} (y-y_0)\, f_\lambda(t)\, dt\\
&\quad+2\int_{((\Delta_\lambda^+(y)\cup\Delta_\lambda^+(y_0))\setminus(\Delta_\lambda^+(y)\cap\Delta_\lambda^+(y_0)))} (s_\lambda^+(t, y)-s_\lambda^+(t, y_0))\, dt, \qquad (2.6)
\end{aligned}$$

where $\Delta_\lambda^\pm(y)$ is the support of $s_\lambda^\pm(t, y)$, respectively. Let $\tilde{y}$ be the infimum of all $y\geqslant 0$ such that $s_\lambda(t, y)$ has at least one zero in $\mathrm{Int}(S_{w^\lambda})$. Since $yf_\lambda(t)$ increases with y, then $\Delta_\lambda^+(y_1)\subseteq\Delta_\lambda^+(y_2)$ for $y_1>y_2\geqslant 0$ and if we assume that for some $y_1>y_2\geqslant\tilde{y}$, $\Delta_\lambda^+(y_1)\equiv\Delta_\lambda^+(y_2)$, then at $t\in\Delta_\lambda^+(y_1)\cap\Delta_\lambda^-(y_1)$ we would have

$$v_\lambda(t)=\lambda y_1\, f_\lambda(t)>\lambda y_2\, f_\lambda(t)=v_\lambda(t)$$

which is impossible. Furthermore, $\Delta_\lambda^+(y)\to\Delta_\lambda^+(y_0)$ in the sense that the Lebesgue measure of the set $(\Delta_\lambda^+(y)\cup\Delta_\lambda^+(y_0))\setminus(\Delta_\lambda^+(y)\cap\Delta_\lambda^+(y_0))$ tends to zero as $y\to y_0$. Otherwise there will be a set E with positive Lebesgue measure and a number $y_0>0$ such that $E\subseteq\Delta_\lambda^+(y)$ for all $y\in[0, y_0)$, but $E\cap\Delta_\lambda^+(y_0)=\varnothing$. Then for $t\in E$ we will have $0\geqslant s_\lambda(t, y_0)=\lim_{y\to y_0} s_\lambda(t, y)\geqslant 0$ hence $s_\lambda(t, y_0)=0$ which contradicts the assumption that $s_\lambda(t, y_0)$ has countably many zeros in S_{w^λ}.

For $t\notin\Delta_\lambda^+(y)\cap\Delta_\lambda^+(y_0)$ we have

$$|s_\lambda^+(t, y)-s_\lambda^+(t, y_0)|\leqslant|s_\lambda(t, y)-s_\lambda(t, y_0)|=|y-y_0|\, f_\lambda(t),$$

and therefore the absolute value of the last integral in (2.6) is at most

$$|y-y_0|\int_{((\Delta_\lambda^+(y)\cup\Delta_\lambda^+(y_0))\setminus(\Delta_\lambda^+(y)\cap\Delta_\lambda^+(y_0)))} f_\lambda(t)\, dt=o(|y-y_0|).$$

Hence from (2.6) we obtain

$$m_\lambda'(y_0) = 1 - 2\int_{\Delta_\lambda^+(y_0)} f_\lambda(t)\,dt. \tag{2.7}$$

Then $m_\lambda'(y) \in C(\mathbf{R})$ follows from (2.7) and the fact that $\Delta_\lambda^+(y)$ continuously changes with y.

For $y \leqslant \tilde{y}$, $\Delta_\lambda^-(y) \equiv \varnothing$ and by (2.7), $m_\lambda'(y) = -1$, and $m_\lambda(y) = p_\lambda(y) = 1/\lambda - y$. For $y > \tilde{y}$, $\Delta_\lambda^+(y)$ decreases with y and by (2.7) we get that $m_\lambda'(y)$ increases on $(\tilde{y}, \infty)$, and $m_\lambda'(y) \to 1$ as $y \to \infty$. Then there is a unique $y^* = y^*(\lambda) > \tilde{y}$ such that $m_\lambda'(y^*) = 0$ and by (1.13), $m_\lambda = m_\lambda(y^*)$. Lemma 2.2 is proved. ▮

Proof of Theorem 1.6. We first show that m_λ is a decreasing function of $\lambda > 0$. Let $\lambda_1 > \lambda > 0$. By (2.5) we have

$$p_{\lambda_1}(y) < p_\lambda(y + 1/\lambda - 1/\lambda_1), \qquad y \geqslant 0.$$

Since $m_\lambda(y) = 2p_\lambda(y) + y - 1/\lambda$, for $y \geqslant 0$ we have

$$m_{\lambda_1}(y) < 2p_\lambda(y + 1/\lambda - 1/\lambda_1) + y - 1/\lambda_1 = m_\lambda(y + 1/\lambda - 1/\lambda_1). \tag{2.8}$$

Then from (1.13) and (2.8) and Lemma 2.2 ($m_\lambda(y) \in C(\mathbf{R})$) we get

$$\begin{aligned} m_{\lambda_1} &= \min\{m_{\lambda_1}(y) \colon y \in [0, 2/\lambda_1]\} \\ &< \min\{m_\lambda(y + 1/\lambda - 1/\lambda_1) \colon y \in [0, 2/\lambda_1]\} \\ &= \min\{m_\lambda(y) \colon y \in [1/\lambda - 1/\lambda_1, 1/\lambda + 1/\lambda_1]\}. \end{aligned} \tag{2.9}$$

By the continuity of $m_\lambda(y)$ (Lemma 2.2) the right-hand side of (2.9) tends to $\min\{m_\lambda(y) \colon y \in [0, 2/\lambda]\} = m_\lambda$ as $\lambda_1 \to \lambda$, $\lambda_1 > \lambda$. Hence m_λ is right-continuous and nondecreasing function of $\lambda > 0$. Now assume that for some $\lambda_2 > \lambda > 0$, $m_{\lambda_2} = m_\lambda$. Then for every $\lambda_1 \in (\lambda, \lambda_2]$, $m_{\lambda_1} = m_\lambda$. Then (2.9) implies that for every $\lambda_1 \in (\lambda, \lambda_2]$, $m_{\lambda_1} = m_\lambda = m_\lambda(y_\lambda)$ for some $y_\lambda \in [0, 1/\lambda - 1/\lambda_1) \cup (1/\lambda + 1/\lambda_1, 2/\lambda]$. By Lemma 2.2 this $y_\lambda = y^*(\lambda)$ is unique, hence $y_\lambda = 0$ or $y_\lambda = 2/\lambda$, that is $m_\lambda'(0) = 0$ or $m_\lambda'(2/\lambda) = 0$. But this is impossible since by (2.7) of Lemma 2.2 and $\Delta_\lambda^+(0) = \operatorname{supp}(\sigma_\lambda^+(0)) = S_{w^\lambda}$ we have $m_\lambda'(0) = -1$, and by (1.12) and $m_\lambda(0) = 1/\lambda$, $y^*(\lambda) = 2/\lambda$ implies $m_\lambda = 1/\lambda$ and $s_\lambda(t, 2/\lambda) \leqslant 0$ on S_{w^λ}, which in view of (2.7) gives $m_\lambda'(2/\lambda) = 1$. Hence m_λ is a decreasing function of $\lambda > 0$.

Now let $\gamma > 0$ be given. First let $E \subset \operatorname{Int}(S_{w^{\lambda(\gamma)}})$ be a compact set. As in the proof of Theorem 1.5 it follows that there is a $\lambda > \lambda(\gamma)$ such that $E \subseteq S_{w^\lambda}$. Moreover,

$$\delta := \gamma - m_\lambda = m_{\lambda(\gamma)} - m_\lambda > 0,$$

and $m_\lambda > 0$ for otherwise $s_\lambda(t, y^*(\lambda)) = 0$ on S_{w^λ} which contradicts the assumption concerning the zeros of the functions $s_\lambda(t, y)$.

Let $a_\lambda := \inf\{y > 0 : n_\lambda(y) > 0\}$, and $b_\lambda := \sup\{y > 0 : p_\lambda(y) > 0\}$. Then $0 \leqslant a_\lambda < b_\lambda \leqslant \infty$, because $p_\lambda(y)$ and $-n_\lambda(y)$ are nonincreasing functions of $y \in \mathbf{R}$, and $m_\lambda > 0$. Moreover, $y^*(\lambda) \in [a_\lambda, b_\lambda]$. Indeed if say $y^*(\lambda) < a_\lambda$, then for every $y \in (y^*(\lambda), a_\lambda)$ we would have $m_\lambda = m_\lambda(y^*(\lambda)) = p_\lambda(y^*(\lambda)) > p_\lambda(y) = m_\lambda(y)$ which contradicts the definition of m_λ. By the continuity of $m_\lambda(y)$ and hence that of $p_\lambda(y) = (m_\lambda(y) - y + 1/\lambda)/2$ and $n_\lambda(y) = m_\lambda(y) - p_\lambda(y)$, we can select $y_0 \in (a_\lambda, b_\lambda)$ with $|p_\lambda(y_0) - p_\lambda(y^*(\lambda))| \leqslant \delta/4$, and $|n_\lambda(y_0) - n_\lambda(y^*(\lambda))| \leqslant \delta/4$. Then we set $\alpha := p_\lambda(y_0) + \delta/8$ and $\beta := n_\lambda(y_0) + \delta/8$. We have $\alpha + \beta \leqslant m_\lambda + 3\delta/4 < \gamma$, $p_\lambda(y_0) < \alpha$, and $n_\lambda(y_0) < \beta$. Hence by Theorem 1.1, every function $f \in C(E)$ is uniformly approximable on E by a sequence of weighted rationals $\{w^n p_n/q_n\}$ with $p_n \in \mathscr{P}_{[\alpha n]}$ and $q_n \in \mathscr{P}_{[\beta n]}$.

Conversely, let $\lambda \in (0, \lambda(\gamma))$. Then $S_{w^{\lambda(\gamma)}} \subset S_{w^\lambda}$, and $m_\lambda > m_{\lambda(\gamma)} = \gamma$. Consider the compact set $E := S_{w^{\lambda(\gamma)}}$. We recall that under the conditions of the theorem E is the union of finitely many closed intervals. Then the constant function 1 on E is not w-approximable in the sense of (A2). Indeed, assume that there are $\alpha \geqslant 0$ and $\beta \geqslant 0$ with $\alpha + \beta \in (0, \gamma]$, and a sequence $\{w^n p_n/q_n\}$ with $p_n \in \mathscr{P}_{[\alpha n]}$ and $q_n \in \mathscr{P}_{[\beta n]}$ that tends to 1 uniformly on E as $n \to \infty$. By Theorem 1.1 there exists $y \in \mathbf{R}$ with $p_\lambda(y) \leqslant \alpha$ and $n_\lambda(y) \leqslant \beta$. Then $m_\lambda \leqslant m_\lambda(y) \leqslant \alpha + \beta \leqslant \gamma$ gives a contradiction. Theorem 1.6 is proved. ∎

3. WEIGHTED RATIONAL APPROXIMATION WITH LAGUERRE AND FREUD WEIGHTS

Laguerre weights. The function $w(u) = u^\theta e^{-cu}$ with $\theta \geqslant 0$ and $c > 0$ defined on $\Sigma = [0, \infty)$ is called *Laguerre weight*. It is known that ([8], Examples IV.1.18 and IV.5.4)

$$S_w = [a(\theta, c), b(\theta, c)] =: \triangle_{\theta, c} \tag{3.1}$$

is an interval with endpoints $a(\theta, c) = 1/c(\theta + 1 - \sqrt{2\theta + 1})$ and $b(\theta, c) = 1/c(\theta + 1 + \sqrt{2\theta + 1})$, and the extremal measure μ_w has density

$$v_w(t) = \frac{c}{\pi t}\sqrt{(t - a(\theta, c))(b(\theta, c) - t)}, \qquad t \in \triangle_{\theta, c}. \tag{3.2}$$

For $\lambda > 0$ we have $w(u)^\lambda = u^{\lambda\theta} e^{-\lambda c u}$, the support $S_{w^\lambda} = \triangle_{\lambda\theta, \lambda c}$,

$$v_\lambda(t) = v_{w^\lambda}(t) = \frac{\lambda c}{\pi t}\sqrt{(t - a)(b - t)}, \qquad t \in \triangle_\lambda := \triangle_{\lambda\theta, \lambda c},$$

where $a = a(\lambda\theta, \lambda c)$ and $b = b(\lambda\theta, \lambda c)$, and

$$f_\lambda(t) = \frac{1}{\pi\sqrt{(t-a)(b-t)}}, \qquad t \in \triangle_\lambda$$

is the equilibrium density for the interval $\triangle_\lambda$.

The approximation problem (A2) *for Laguerre weights.* Let $\gamma > 0$ be given. To determine m_λ for $\lambda > 0$ we consider the equation

$$v_\lambda(t) = \lambda y f_\lambda(t),$$

which is equivalent to

$$c(t-a)(b-t) = yt \qquad \text{or} \qquad ct^2 + t(y - c(a+b)) + cab = 0. \tag{3.3}$$

The formulas for v_λ and f_λ show that (3.3) has two real solutions $t_{1,2}(y) \in [a, b]$,

$$t_{1,2}(y) = \frac{c(a+b) - y \pm \sqrt{(c(a+b)-y)^2 - 4c^2ab}}{2c} \tag{3.4}$$

if and only if $y \in [0, c(\sqrt{b} - \sqrt{a})^2] = [0, 2/\lambda]$. For other y we have $m_\lambda(y) > m_\lambda$. By Lemma 2.2 $m_\lambda = m_\lambda(y^*)$, where y^* is the unique solution of the equation

$$\int_{t_2(y^*)}^{t_1(y^*)} f_\lambda(t)\, dt = \tfrac{1}{2}. \tag{3.5}$$

Changing variables $t = (a+b)/2 + s(b-a)/2$ in (3.5) we obtain

$$\sin^{-1}(a_1 + a_2) - \sin^{-1}(a_1 - a_2) = \pi/2, \tag{3.6}$$

where

$$a_1 = \frac{-y^*}{c(b-a)} \qquad \text{and} \qquad a_2 = \frac{\sqrt{(c(a+b)-y^*)^2 - 4c^2ab}}{c(b-a)}.$$

We apply the cosine function to both sides of the last equation and simplify to obtain

$$|\sqrt{(1-(a_1+a_2)^2)(1-(a_1-a_2)^2)}| = |a_1^2 - a_2^2|.$$

Simplifying further we obtain $2(a_1^2 + a_2^2) = 1$, or equivalently

$$\frac{y^{*2} + ((c(a+b)-y^*)^2 - 4c^2ab)}{c^2(b-a)^2} = \frac{1}{2},$$

which reduces to

$$4y^{*2} - 4c(a+b)\, y^* + c^2(b-a)^2 = 0.$$

The solutions of the last equation are $y^*_{1,2} = c(a+b \pm 2\sqrt{ab})/2$, and since $a+b = 2(\lambda\theta+1)/(\lambda c)$ and $\sqrt{ab} = \theta/c$ (see (3.1)), we have

$$y_2^* = 1/\lambda \qquad \text{and} \qquad y_1^* = (2\lambda\theta+1)/\lambda.$$

Since the range of $\sin^{-1}$ is $[-\pi/2, \pi/2]$, Eq. (3.6) implies that $a_1 + a_2 \geqslant 0$ which is equivalent to $y^* \leqslant (2\lambda\theta+1)/(\lambda(\lambda\theta+1))$ and y_2^* only satisfies this condition, unless $\theta = 0$ in which case $y_1^* = y_2^*$. Hence,

$$y^* = y_2^* = c(a+b-2\sqrt{ab})/2 = 1/\lambda. \tag{3.7}$$

Next we derive a formula for $m_\lambda(y)$ for $y \in [0, 2/\lambda]$. We have $p_\lambda(0) = 1/\lambda$ and since $m_\lambda(y) = 2p_\lambda(y) + y - 1/\lambda$,

$$p'_\lambda(y) = (m'_\lambda(y) - 1)/2 = -\int_{t_2(y)}^{t_1(y)} f_\lambda(t)\, dt, \tag{3.8}$$

where we used (2.7). Then with

$$s_{1,2}(y) := (2t_{1,2}(y) - a - b)/(b-a)$$

we obtain

$$p'_\lambda(y) = (\sin^{-1}(s_2(y)) - \sin^{-1}(s_1(y)))/\pi. \tag{3,9}$$

Then

$$p_\lambda(y) = 1/\lambda + \int_0^y p'_\lambda(u)\, du = 1/\lambda + (J_2(y) - J_1(y))/\pi$$

and

$$m_\lambda(y) = 1/\lambda + y + 2(J_2(y) - J_1(y))/\pi, \tag{3.10}$$

where

$$\begin{aligned} J_{1,2}(y) &:= \int_0^y \sin^{-1}(s_{1,2}(u))\, du \\ &= y \sin^{-1}(s_{1,2}(y)) - \int_0^y \frac{u s'_{1,2}(u)}{\sqrt{1 - s_{1,2}(u)^2}}\, du \\ &= y \sin^{-1}(s_{1,2}(y)) - \int_0^y \frac{u t'_{1,2}(u)}{\sqrt{(t_{1,2}(u) - a)(b - t_{1,2}(u))}}\, du. \end{aligned} \tag{3.11}$$

For $t_{1,2}(u)$ from (3.4) we get

$$t'_{1,2}(u) = \frac{\mp t_{1,2}(u)}{c(t_1(u) - t_2(u))} \tag{3.12}$$

and by (3.3)

$$\sqrt{(t_{1,2}(u) - a)(b - t_{1,2}(u))} = \sqrt{u t_{1,2}(u)/c}. \tag{3.13}$$

Then by (3.11), (3.12) and (3.13) we obtain

$$\begin{aligned} J_2(y) - J_1(y) &= y(\sin^{-1}(s_2(y)) - \sin^{-1}(s_1(y))) \\ &\quad - \frac{1}{\sqrt{c}} \int_0^y \frac{\sqrt{u}(\sqrt{t_1(u)} + \sqrt{t_2(u)})}{t_1(u) - t_2(u)}\, du. \end{aligned}$$

For the last integral we have by (3.3)

$$\begin{aligned} \int_0^y \frac{\sqrt{u}}{\sqrt{t_1(u)} - \sqrt{t_2(u)}}\, du &= \int_0^y \frac{\sqrt{u}}{\sqrt{t_1(u) + t_2(u) - 2\sqrt{t_1(u)\, t_2(u)}}}\, du \\ &= \int_0^y \frac{\sqrt{u}}{\sqrt{(a+b) - u/c - 2\sqrt{ab}}}\, du \\ &= \sqrt{c} \int_0^y \frac{\sqrt{u}}{\sqrt{2/\lambda - u}}\, du =: \frac{\sqrt{c}}{\lambda} A(\lambda y). \end{aligned}$$

To compute $A(y)$ we use change of variables $u \to 2v^2$ and integration by parts

$$\begin{aligned} A(y) &= 4 \int_0^{\sqrt{y/2}} \frac{v^2}{\sqrt{1-v^2}}\, dv = -4 \int_0^{\sqrt{y/2}} \sqrt{1-v^2}\, dv \\ &\quad + 4 \int_0^{\sqrt{y/2}} \frac{1}{\sqrt{1-v^2}}\, dv = -4 \sqrt{\frac{y}{2}\left(1 - \frac{y}{2}\right)} - A(y) + 4 \sin^{-1}(\sqrt{y/2}), \end{aligned}$$

and so we obtain

$$A(y) = 2 \sin^{-1}(\sqrt{y/2}) - \sqrt{y(2-y)}. \tag{3.14}$$

Then

$$J_2(y) - J_1(y) = y(\sin^{-1}(s_2(y)) - \sin^{-1}(s_1(y))) - A(\lambda y)/\lambda$$

and for $m_\lambda(y)$ from (3.8), (3.9), and (3.10) we obtain

$$m_\lambda(y) = 1/\lambda + y - 2y \int_{t_2(y)}^{t_1(y)} f_\lambda(t)\, dt - 2A(\lambda y)/(\lambda\pi). \tag{3.15}$$

For the minimal mass m_λ we get (using (3.5) and (3.14))

$$\begin{aligned} m_\lambda = m_\lambda(y^*) = m_\lambda(1/\lambda) &= 2/\lambda - (2/\lambda) \int_{t_2(1/\lambda)}^{t_1(1/\lambda)} f_\lambda(t)\, dt \\ &- 2A(1)/(\lambda\pi) = 1/\lambda - 2(\pi/2 - 1)/(\lambda\pi) = 2/(\lambda\pi). \end{aligned} \tag{3.16}$$

The quantity m_λ decreases from ∞ to m_1 as λ increases from 0 to 1. Then by Theorem 1.6 for a given $\gamma \geqslant m_1$ the largest interval $\triangle_\lambda := \triangle_{\lambda\theta,\, \lambda c}$ on which approximation by weighted rationals is possible in the sense of (A2) is the interval $\triangle_{\lambda(\gamma)}$, where $\lambda(\gamma) = 2/(\pi\gamma)$.

Freud weights. The function $w(u) = \exp(-\gamma_\tau\, |u|^\tau)$, with $\tau > 0$ and

$$\gamma_\tau = \frac{\Gamma(\tau/2)\, \Gamma(1/2)}{2\Gamma((\tau+1)/2)},$$

defined on $\Sigma = \mathbf{R}$ is called *Freud weight*. By [8, Theorem IV.5.1], $S_w = [-1, 1]$ and $\mu_{w_\tau}(t) = s_\tau(t)\, dt$, where

$$s_\tau(t) = \frac{\tau}{\pi} \int_{|t|}^1 \frac{u^{\tau-1}}{\sqrt{u^2 - t^2}}\, du, \qquad t \in [-1, 1] \tag{3.17}$$

is the so called *Ullman distribution.*

The approximation problem (A2) *for Freud weights.* Let $\lambda > 0$. For $w(u)^\lambda = \exp(-\lambda\gamma_\tau\, |u|^\tau)$ it follows from the definition of the extremal measure that $S_{w^\lambda} = [-\lambda^{-1/\tau}, \lambda^{-1/\tau}] =: \triangle_\lambda$, and

$$v_\lambda(t) = v_{w^\lambda}(t) = s_\tau(\lambda^{1/\tau} t)\, \lambda^{1/\tau}, \qquad t \in \triangle_\lambda.$$

The function s_τ is even and as we are going to show later with Lemma 3.3, for $\tau \in [1, 2]$, $s_\tau(t)$ is monotone decreasing on $[0, 1]$ and so is $v_\lambda(t)$ on $[0, \lambda^{-1/\tau}]$.

We shall restrict ourselves to Freud weights with $\tau \in [1, 2]$ since in this case the monotonicity of s_τ allows us to solve the problem completely. For $y \geqslant 0$ we consider the function

$$s_\lambda(t, y) = (1/\lambda)\, v_\lambda(t) - y f_\lambda(t), \qquad t \in \triangle_\lambda,$$

where

$$f_\lambda(t)=\frac{1}{\pi\sqrt{\lambda^{-2/\tau}-t^2}}, \qquad t\in\triangle_\lambda$$

is the equilibrium density for $\triangle_\lambda$. The equation $s_\lambda(t, y)=0$ has exactly two solutions $t_1(y)>0$ and $t_2(y)=-t_1(y)$ in $\triangle_\lambda$ for $y\in[0, a_{\tau,\lambda})$, where

$$a_{\tau,\lambda}:=\frac{v_\lambda(0)}{\lambda f_\lambda(0)}=\frac{\tau}{\lambda(\tau-1)}.$$

By the proof of Theorem 1.6 and Lemma 2.2 we have

$$m_\lambda=\min\{m_\lambda(y)\colon y\in[0, a_{\tau,\lambda}]\}=m_\lambda(y^*),$$

where $y^*\in[0, a_{\tau,\lambda})$ is the unique solution of the equation

$$\frac{1}{2}=\int_{-t_1(y)}^{t_1(y)} f_\lambda(t)\,dt=\frac{2}{\pi}\sin^{-1}(\lambda^{1/\tau}t_1(y)).$$

Then $\lambda^{1/\tau}t_1(y^*)=\sqrt{2}/2$, and for m_λ we obtain

$$\begin{aligned} m_\lambda=m_\lambda(y^*)&=2p_\lambda(y^*)+y^*-\frac{1}{\lambda}\\ &=\frac{2}{\lambda}\int_{-t_1(y^*)}^{t_1(y^*)} v_\lambda(t)\,dt-2y^*\int_{-t_1(y^*)}^{t_1(y^*)} f_\lambda(t)\,dt+y^*-\frac{1}{\lambda}\\ &=\frac{2}{\lambda}\int_{-t_1(y^*)}^{t_1(y^*)} v_\lambda(t)\,dt-\frac{1}{\lambda}=\frac{4}{\lambda}\int_0^{\sqrt{2}/2} s_\tau(u)\,du-\frac{1}{\lambda}. \end{aligned} \tag{3.18}$$

To compute the last integral we need a differential equation for $s_\tau(t)$. Let $t\in(0, 1)$. With the change of variables $u\to tu_1$ and $u_1\to 1/u$ we obtain

$$\begin{aligned} s_\tau(t)&=\frac{\tau}{\pi}\int_t^1 \frac{u^{\tau-1}}{\sqrt{u^2-t^2}}\,du\\ &=\frac{\tau}{\pi}\,t^{\tau-1}\int_1^{1/t}\frac{u_1^{\tau-1}}{\sqrt{u_1^2-1}}\,du_1=\frac{\tau}{\pi}\,t^{\tau-1}\int_t^1\frac{u^{-\tau}}{\sqrt{1-u^2}}\,du. \end{aligned} \tag{3.19}$$

Then

$$s_\tau'(t)=\frac{\tau}{\pi}\left((\tau-1)\,t^{\tau-2}\int_t^1\frac{u^{-\tau}}{\sqrt{1-u^2}}\,du-t^{\tau-1}\frac{t^{-\tau}}{\sqrt{1-t^2}}\right)$$

or equivalently

$$ts'_\tau(t) = (\tau - 1)\, s_\tau(t) - \frac{\tau}{\pi\sqrt{1-t^2}}. \tag{3.20}$$

For $a \in [0, 1]$ using integration by parts and (3.20) we obtain

$$\begin{aligned} I_\tau(a) :&= \int_0^a s_\tau(t)\, dt = as_\tau(a) - \int_0^a ts'_\tau(t)\, dt \\ &= as_\tau(a) + \frac{\tau}{\pi} \sin^{-1}(a) - (\tau - 1)\, I_\tau(a), \end{aligned}$$

hence

$$I_\tau(a) = \frac{a}{\tau} s_\tau(a) + \frac{1}{\pi} \sin^{-1}(a). \tag{3.21}$$

From (3.18) and (3.21) we obtain

$$m_\lambda = \frac{4I_\tau(\sqrt{2}/2) - 1}{\lambda} = \frac{2\sqrt{2}}{\tau\lambda} s_\tau(\sqrt{2}/2). \tag{3.22}$$

Then m_λ decreases from ∞ to m_1 as λ increases from 0 to 1. By Theorem 1.6, for given $\gamma \geqslant m_1$ the largest interval $\triangle_\lambda$ on which weighted rational approximation is possible in the sense of (A2) is the interval $\triangle_{\lambda(\gamma)}$, where (see (1.14))

$$\lambda(\gamma) = \frac{2\sqrt{2}}{\tau\gamma} s_\tau(\sqrt{2}/2). \tag{3.23}$$

The approximation problem (A1) *for Freud weights.* Let $\alpha \geqslant 0$ and $\beta \geqslant 0$ with $\alpha + \beta > 0$ be given. The Freud weights satisfy the conditions of Theorem 1.5, hence by Theorem 1.5 we have

$$\lambda^*(\alpha, \beta) = \inf \{ \lambda > 0 : \exists y \in \mathbf{R} : h_\lambda(y) \leqslant \alpha \}.$$

As shown in the proof of Theorem 1.5, for every $\lambda > 0$ the equation $p_\lambda(y) = (\alpha/\beta)\, n_\lambda(y)$ has unique solution $\bar{y}(\alpha, \beta; \lambda) > 0$. Moreover, by the proof of Theorem 1.5 it follows that $\lambda^*(\alpha, \beta)$ is the unique solution of the equation

$$p_\lambda(\bar{y}(\alpha, \beta; \lambda)) = \alpha.$$

For $t \in \triangle_\lambda$ we have $u := \lambda^{1/\tau} t \in [-1, 1]$ and

$$s_\lambda(t, y) = \lambda^{1/\tau}\left(\frac{1}{\lambda}\, s_\tau(u) - \frac{y}{\pi\sqrt{1-u^2}}\right) =: \lambda^{1/\tau - 1}\tilde{s}(u, \lambda y).$$

Then with $\tilde{p}(y) = \|\tilde{s}^+(u, y)\|$ and $\tilde{n}(y) = \|\tilde{s}^-(u, y)\|$ we have $p_\lambda(y) = \lambda^{-1}\tilde{p}(\lambda y)$ and $n_\lambda(y) = \lambda^{-1}\tilde{n}(\lambda y)$. Moreover, $\bar{y}(\alpha, \beta; \lambda) = \lambda^{-1}\tilde{y}(\alpha, \beta)$, where $\tilde{y}(\alpha, \beta)$ is the unique solution of the equation $\tilde{p}(y) = (\alpha/\beta)\, \tilde{n}(y)$. Hence $\lambda^*(\alpha, \beta)$ is the unique solution of the equation $p_\lambda(\lambda^{-1}\tilde{y}(\alpha, \beta)) = \alpha$, that is, $\tilde{p}(\tilde{y}(\alpha, \beta)) = \lambda\alpha$. Therefore

$$\lambda^*(\alpha, \beta) = \tilde{p}(\tilde{y}(\alpha, \beta))/\alpha. \tag{3.24}$$

Here $\tilde{y}(\alpha, \beta) = \tilde{y}(\tau; \alpha, \beta)$ and $\lambda^*(\alpha, \beta) = \lambda^*(\tau; \alpha, \beta)$ depend on τ as well.

Now let $\tau \in [1, 2]$. In this case by Lemma 3.3 for $y \in (0, a_\tau)$ the equation $\tilde{s}(u, y) = 0$ has exactly two solutions $u_1(y) > 0$ and $u_2(y) = -u_1(y)$ in $(-1, 1)$, where $a_\tau := \sup\{y > 0 : \tilde{p}(y) > 0\}$. Then

$$\tilde{p}(y) = 2\int_0^{u_1(y)} s_\tau(t)\, dt - (2y/\pi)\sin^{-1}(u_1(y)).$$

From (3.21) for the last integral we obtain

$$I_\tau(u_1(y)) = (1/\tau)\, u_1(y)\, s_\tau(u_1(y)) + (1/\pi)\sin^{-1}(u_1(y)),$$

hence

$$\tilde{p}(y) = (2/\tau)\, u_1(y)\, s_\tau(u_1(y)) + (2/\pi)(1-y)\sin^{-1}(u_1(y)). \tag{3.25}$$

On the other hand using that $\tilde{p}(y) - \tilde{n}(y) = 1 - y$ we can write the equation $\tilde{p}(y) = (\alpha/\beta)\, \tilde{n}(y)$ in the form $(\beta - \alpha)\, \tilde{p}(y) = \alpha(y-1)$. If $\alpha \neq \beta$ by (3.24) we get

$$\lambda^*(\tau; \alpha, \beta) = \frac{\tilde{y}(\tau; \alpha, \beta) - 1}{\beta - \alpha}. \tag{3.26}$$

If $\alpha = \beta$ then $\tilde{y}(\tau; \alpha, \alpha) = 1$ and by (3.24) and (3.25),

$$\lambda^*(\tau; \alpha, \alpha) = 2u_1(1)\, s_\tau(u_1(1))/(\alpha\tau). \tag{3.27}$$

We now consider the special case $\tau = 2$. We have $s_2(t) = (2/\pi)\sqrt{1-t^2}$ (see (3.17)) and solving $\tilde{s}(u, y) = 0$ we get $u_{1,2}(y) = \pm\sqrt{1 - y/2}$ for $y \in [0, 2)$. Hence by (3.25) we get that $\tilde{y}(2; \alpha, \beta)$ is the solution of the equation

$$(1/\pi)(\beta - \alpha)(\sqrt{y(2-y)} + 2(1-y)\sin^{-1}(\sqrt{1-y/2})) = \alpha(y-1). \tag{3.28}$$

Then $\lambda^*(2; \alpha, \beta) = (\tilde{y}(2; \alpha, \beta) - 1)/(\beta - \alpha)$ if $\alpha \neq \beta$, and

$$\lambda^*(2; \alpha, \alpha) = \frac{\tilde{p}(\tilde{y}(2; \alpha, \alpha))}{\alpha} = \frac{\tilde{p}(1)}{\alpha} = \frac{1}{\alpha\sqrt{2}}\, s_2\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\pi\alpha}.$$

Next we show that the Ullman distribution s_τ for $\tau \in [1, 2]$ is monotone on $[0, 1]$.

LEMMA 3.3. *For every $\tau \in [1, 2]$ the Ullman distribution s_τ is a monotone decreasing function on the interval $[0, 1]$.*

Proof. First let $\tau \in (1, 2]$. We will show that $s_\tau'(t) < 0$ on $(0, 1)$ which in view of (3.20) is equivalent to

$$s_\tau(t) < \frac{\tau}{\pi(\tau - 1)\sqrt{1 - t^2}}, \qquad t \in (0, 1),$$

or using (3.19) it is the same as

$$t^{\tau - 1} \int_t^1 \frac{u^{-\tau}}{\sqrt{1 - u^2}}\, du < \frac{1}{(\tau - 1)\sqrt{1 - t^2}}, \qquad t \in (0, 1). \tag{3.29}$$

For $u \in [0, 1)$ we have the power series expansion

$$(1 - u)^{-1/2} = \sum_{k=0}^{\infty} (-1)^k \binom{-1/2}{k} u^k =: \sum_{k=0}^{\infty} c_k u^k,$$

where $c_0 = 1$ and

$$c_k = \frac{(2k - 1)!!}{k!\, 2^k} = O(k^{-1/2})$$

for large $k \in \mathbf{N}$. Then (3.29) is equivalent to each of the following

$$t^{\tau - 1} \sum_{k=0}^{\infty} c_k \int_t^1 u^{2k - \tau}\, du < \frac{1}{(\tau - 1)} \sum_{k=0}^{\infty} c_k t^{2k},$$

$$t^{\tau - 1} \sum_{k=0}^{\infty} c_k \left(\frac{1 - t^{2k + 1 - \tau}}{2k + 1 - \tau}\right) < \frac{1}{(\tau - 1)} \sum_{k=0}^{\infty} c_k t^{2k},$$

and

$$\frac{t^{\tau - 1}}{(\tau - 1)} > \sum_{k=1}^{\infty} c_k \left(\frac{t^{\tau - 1}}{(2k + 1 - \tau)} - \frac{2k t^{2k}}{(\tau - 1)(2k + 1 - \tau)}\right), \qquad t \in (0, 1).$$

The last inequality follows from

$$\frac{1}{(\tau-1)} \geqslant \sum_{k=1}^{\infty} c_k \frac{1}{(2k+1-\tau)}, \qquad \tau \in (1, 2]. \tag{3.30}$$

To verify (3.30) we consider the function

$$F(\tau) = \frac{1}{(\tau-1)} - \sum_{k=1}^{\infty} c_k \frac{1}{(2k+1-\tau)}, \qquad \tau \in (1, 2].$$

We have $F(\tau) \to \infty$ as $\tau \to 1^+$ and

$$F'(\tau) = -(\tau-1)^{-2} - \sum_{k=1}^{\infty} c_k (2k+1-\tau)^{-2} < 0, \qquad \tau \in (1, 2].$$

So it is enough to show that $F(2) \geqslant 0$. Using the same expansion as before we obtain

$$\int_t^1 \frac{u^{-2}}{\sqrt{1-u^2}}\, du = \sum_{k=0}^{\infty} c_k \left(\frac{1-t^{2k-1}}{2k-1} \right), \qquad t \in (0, 1)$$

which implies

$$\sum_{k=1}^{\infty} c_k \frac{1}{(2k-1)} = 1 + \sum_{k=1}^{\infty} c_k \frac{t^{2k-1}}{(2k-1)} + \int_t^1 \frac{u^{-2}}{\sqrt{1-u^2}}\, du - \frac{1}{t}$$

for $t \in (0, 1)$. Next for $t \in (0, 1)$ from (3.19) we get

$$\int_t^1 \frac{u^{-2}}{\sqrt{1-u^2}}\, du - \frac{1}{t} = \frac{1}{t} \left(\frac{\pi}{2} s_2(t) - 1 \right) = \frac{-t}{\sqrt{1-t^2}+1}.$$

Taking a limit as $t \to 0^+$ in the last two equations we obtain

$$F(2) = 1 - \sum_{k=1}^{\infty} c_k \frac{1}{(2k-1)} = 0.$$

For $\tau = 1$ by (3.17) we get

$$s_1(t) = (1/\pi)(\ln(1/t) + \ln(1+\sqrt{1-t^2})), \qquad t \in (0, 1],$$

a decreasing function on $(0, 1]$. This completes the proof of Lemma 3.3. ∎

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